JHEP HYPER VERSION

Received: *March 4, 1998,* Accepted: *May 7, 1998*

Central functions and their physical implications

Damiano Anselmi
Centre de Physique Theorique, Ecole Polytechnique, F-91128 Palaiseau Cedex, France

Abstract: Central functions $c(g)$ and $c'(g)$ are constructed in quantum field theory. These quantities justify and generalize the notions of central charges recently introduced at criticality, which, together with suitable anomalous dimensions h, identify a conformal field theory in four dimensions (CFT_4). They are encoded in the four-point function of the stress-energy tensors. The behavior of the central functions is analysed to two-loops in perturbation theory. The central function is the fundamental notion for a description of quantum field theory as a radiative interpolation between pairs of CFT_4's. The problem of computating their RG flow in the far IR limit starting from the UV fixed point is addressed in the context of supersymmetric gauge theories and electric-magnetic duality.

Keywords: Renormalization Regularization and Renormalons, Anomalies in Field and String Theories, Supersymmetry and Duality, Conformal and W Symmetry.

JHEP05(1998)005

Contents

The fundamental properties of conformal field theories in four dimensions (CFT_4) are encoded into certain *central charges* c, c' and anomalous dimensions h. The primary central charge c is the coefficient of the square of the Weyl tensor in the expression of the trace anomaly (in presence of an external gravitational field). Secondary central charges c' were defined in ref. [1], motivated by the observation that the OPE of the stress-energy tensor T with itself in general does not close [2]. While in two dimensions the TT OPE contains only the operators 1, T and the derivatives of T, in higher dimensions other operators Σ appear and h denotes their anomalous dimensions while c' is the constant defined by the $\Sigma\Sigma$ two-point function. The operators Σ are, in general, infinitely many, so that a CFT_4 is identified by a primary central charge c and infinitely many secondary central charges c' and anomalous dimensions h. Two operators Σ are noticeable, the mass operator $\bar{\varphi}\varphi$ for scalar fields and the axial current J_μ^5 for fermionic fields: in the TT OPE these operators appear between the operator 1 and T itself:

$$T(x)\,T(0) \sim \frac{c}{|x|^8} + \frac{\bar{\varphi}\varphi}{|x|^{6-h_1}} + \frac{J_\mu^5}{|x|^{5-h_2}} + \frac{T}{|x|^4} + \cdots$$

In a supersymmetric theory $\bar{\varphi}\varphi$ and J_μ^5 are part of a unique superfield, the Konishi current. The first radiative corrections to c and c' were computed in ref. [3]. The result suggested that the central charges are invariant under continuous deformations of CFT_4's. Computations of h can be found both in [1] and [3]. Moreover, in [3] it was noted that to the second loop order in perturbation theory, it is possible to extend the analysis off-criticality and study the first terms of quantities that what we can call *central functions*, which are the main subject of the present paper.

We are interested in quantum field theories that admit well-defined UV and IR fixed points. The existence of natural and, to some extent, unique functions[1] $c(g)$ and $c'(g)$ that interpolate between the critical values of the central charges will be proved by a detailed renormalization group analysis of the correlator of two and four stress-energy tensors. This investigation provides a description of quantum field theory as a

[1]Here g denotes generically the set of coupling constants of the theory.

radiative interpolation between pairs of conformal field theories. Computing the flow of the central functions would allow us to identify the IR fixed point of a quantum field theory starting from the UV one (or vice-versa). Precisely, the central functions $c(g)$ and $c'(g)$ define the flow of the numbers of vector, spinor and scalar degrees of freedom from the UV conformal fixed point to the IR conformal fixed point of a generic quantum field theory.

A “good” central function has first of all to be physical, which means that it should be identified by means of a physical correlator (this, in particular, assures scheme independence: for example, the beta-function is not a good central function). Secondly, it has to admit finite UV and IR limits, between which it interpolates. Third, it has to be a function of the couplings g at a *single* energy scale: this assures that the central function interpolates between UV and IR values in a *unique* way. Many functions have to be excluded because they do not share this property: we shall present concrete examples along with the discussion.

In some special models it might be possible to work out all-order expressions of the central functions, on the same footing, for example, as the NSVZ beta-function [4] of supersymmetric QCD. The investigation of four dimensional conformal field theories from the point of view that we offer here might be fruitful in the context of electric-magnetic duality [5] which is indeed the conjectured equivalence of two SCFT$_4$'s, IR fixed points of quite different quantum field theories. The values of the central charges of equivalent CFT$_4$'s should obviously coincide.[2]

The primary central function $c(g)$ is related to the two-point correlator $\langle T_{\mu\nu}(x)\, T_{\rho\sigma}(y)\rangle$, while the secondary central functions $c'(g)$ are related to channels of the four-point correlator $\langle T_{\mu\nu}(x)\, T_{\rho\sigma}(y)\, T_{\alpha\beta}(z)\, T_{\gamma\delta}(w)\rangle$. It turns out that it is precisely the correlator of four stress tensors that contains the relevant information for a correct description of CFT$_4$'s and quantum field theories interpolating among them. This correlator has no unique form, not even at criticality, but a suitable limiting procedure is able to single out certain universal properties.

We are now ready to identify the central functions inside the two-point and four-point correlators of the stress tensor.

1. Definition of the first central function

In this section we study several properties of the two-point function of the stress tensor. Some observations have already appeared in the literature and are well-known. We focus our attention on the properties that are new and relevant to our purpose, that will be derived in full detail. Other remarks are also meant to introduce the more involved notion of secondary central function (see next section) and illustrate the general procedure, in order to simplify the presentation of the more general case. Applied

[2] *Note added.* Indeed, see paper [11].

JHEP05(1998)005

to two dimensions, our observations allow us to give a very straightforward proof of the c-theorem.

We would like to show that the correlator of two stress-energy tensors can always be written in the form

$$\langle T_{\mu\nu}(x)\, T_{\rho\sigma}(0)\rangle = -\frac{1}{48\pi^4}\Pi_{\mu\nu\rho\sigma}\left(\frac{c(g(x))}{x^4}\right) + \pi_{\mu\nu}\pi_{\rho\sigma}\left(\frac{f(\ln x\mu, g))}{x^4}\right). \tag{1.1}$$

where $\pi_{\mu\nu} = (\partial_\mu\partial_\nu - \delta_{\mu\nu}\Box)$ and $\Pi_{\mu\nu\rho\sigma} = 2\pi_{\mu\nu}\pi_{\rho\sigma} - 3(\pi_{\mu\rho}\pi_{\nu\sigma} + \pi_{\mu\sigma}\pi_{\nu\rho})$. μ is the subtraction point and $g(x)$ is the running coupling constant, having value g at the scale μ. $T_{\mu\nu}$ is normalized as $e^{-1}e^a_\mu\, \delta S/\delta e^\nu_a$, S being the action and e^a_μ the vielbein.

The decomposition (1.1) into a spin-2 term and a spin-0 term is well-known (see for example [6]) and we recall it briefly. The most general expression for $\langle T_{\mu\nu}(x)\, T_{\rho\sigma}(0)\rangle$ is the sum of five independent tensors constructed with x_α and $\delta_{\alpha\beta}$. On the other hand, $\partial_\mu\langle T_{\mu\nu}(x)\, T_{\rho\sigma}(0)\rangle$ contains only three independent terms. Therefore, only two independent functions survive after imposing the conservation condition $\partial_\mu\langle T_{\mu\nu}(x)\, T_{\rho\sigma}(0)\rangle = 0$, which holds at any x different from zero. We denote these two functions by c (spin two) and f (spin zero).

Our task is to show that the function here denoted by $c(g)$ is indeed just a function of the running coupling g (this in not necessarily true for the function f), which allows us to conclude that $c(g)$ is a good central function, while f is not.

This is due to the fact that in general the stress tensor is not a truly finite operator [7]. Invariance under translations assures that its divergence $\partial_\mu T_{\mu\nu}$ is a finite operator, so that we can write

$$\partial_\mu T^R_{\mu\nu} = \partial_\mu T^B_{\mu\nu}\,, \tag{1.2}$$

but not necessarily that $T_{\mu\nu}$ itself is finite (the suffixes R and B denote the renormalized and bare operators, respectively). Indeed, (1.2) fails to imply $T^R_{\mu\nu} = T^B_{\mu\nu}$ when scalar fields ϕ are present, since we can always write down an expression of the form

$$T^R_{\mu\nu} = T^B_{\mu\nu} + A\pi_{\mu\nu}\phi^2\,, \tag{1.3}$$

A being an arbitrary constant. The difference between the bare and renormalized stress tensors shown in (1.3) is fixed uniquely by locality, power counting and conservation.

Therefore, in order to apply the Callan-Symanzik equations, one has to consider the couple of operators $(\mathcal{S}_1, \mathcal{S}_2) \equiv (T_{\mu\nu}, \pi_{\mu\nu}\phi^2)$. These two operators will mix at the quantum level (which means in particular that A can be divergent) and one has to study the matrix of correlators $\langle \mathcal{S}_i(x)\, \mathcal{S}_j(0)\rangle$, together with the matrix of renormalization constants

$$Z_{ij} = \begin{pmatrix} 1 & A \\ 0 & Z_{\phi^2}\end{pmatrix}, \tag{1.4}$$

Z_{ϕ^2} being the renormalization constant of the mass operator ϕ^2. The Callan-Symanzik equations are then

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g}\right)\langle \mathcal{S}_i(x)\, \mathcal{S}_i(0)\rangle = \gamma_{ik}\langle \mathcal{S}_k(x)\, \mathcal{S}_j(0)\rangle + \langle \mathcal{S}_i(x)\, \mathcal{S}_k(0)\rangle\gamma_{kj}\,, \tag{1.5}$$

with $\gamma_{ij} = dZ_{ik}/d\ln\mu\, Z^{-1}_{kj}$.

While $\langle \mathcal{S}_1(x)\, \mathcal{S}_1(0)\rangle$ can be written as a sum of the two terms (1.1), it is clear that in all the other cases we can write

$$\langle \mathcal{S}_i(x)\, \mathcal{S}_j(0)\rangle = \pi_{\mu\nu}\pi_{\rho\sigma}\left(\frac{f_{ij}(\ln x\mu, g))}{x^4}\right) \tag{1.6}$$

for $(i,j) \neq (1,1)$: there cannot be any spin-2 term, but only a spin-0 term.

Therefore, one can immediately see that the mixing between $\mathcal{S}_1$ and $\mathcal{S}_2$ affects only the function f and not c. For c we can write

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g}\right) c(g, \ln x\mu) = 0\,, \tag{1.7}$$

which concludes the proof that $c(g, \ln x\mu) = c(g(x))$.

We can further illustrate our point by introducing a nonlocal traceless stress energy tensor $\mathcal{T}_{\mu\nu}$ defined as

$$\mathcal{T}_{\mu\nu} = T_{\mu\nu} + \frac{1}{3}\frac{1}{\Box}\pi_{\mu\nu}T\,,$$

where $T = T_{\alpha\alpha}$. Indeed $\mathcal{T}_{\mu\nu}$ satisfies

$$\langle \mathcal{T}_{\mu\nu}(x)\, \mathcal{T}_{\rho\sigma}(0)\rangle = -\frac{1}{48\pi^4}\Pi_{\mu\nu\rho\sigma}\left(\frac{c(g(x))}{x^4}\right), \tag{1.8}$$

so that we can formally write $\mathcal{T}^R_{\mu\nu} = \mathcal{T}^B_{\mu\nu}$. This remark will be used in the next section in order to eliminate the functions of the type f from the very beginning and simplify the derivation of the notion of secondary central function. $\mathcal{T}_{\mu\nu}$ should be considered not as a fundamental operator with which to work out operator product expansions, but only as a convenient way to collect the information, once it has been worked out in the usual way. When this is kept in mind, the nonlocality of $\mathcal{T}_{\mu\nu}$ is under control.

The function $c(g)$ appearing in (1.1) or (1.8) can be taken as the all-order definition of primary central function. $c(g)$ correctly interpolates between the values of the central charges at the critical points. Moreover, it coincides with the definition used in [3] to the second loop order in perturbation theory around a free conformal fixed point.

Before proceeding, we have to make a final remark, necessary for completeness. In applying the Callan-Symanzik equations we have assumed that the equations satisfied by $\Pi_{\mu\nu\rho\sigma}(c/x^4)$ and $\pi_{\mu\nu}\pi_{\rho\sigma}(f/x^4)$ extend directly to the functions c and f (see (1.7) in particular). The reason is that we can safely corss the forth-order differential operators $\Pi_{\mu\nu\rho\sigma}$ and $\pi_{\mu\nu}\pi_{\rho\sigma}$. When passing through these operators, one has to worry about possible ambiguities of the form $c \to c + |x|^4\mathcal{P}_4(x)$, $f \to f + |x|^4\mathcal{P}'_4(x)$, $\mathcal{P}_4(x)$, $\mathcal{P}'_4(x)$ being polynomials of order less than four (further restricted by Lorentz invariance to the form $a + bx^2$), which are indentically annihilated by $\Pi_{\mu\nu\rho\sigma}$ and $\pi_{\mu\nu}\pi_{\rho\sigma}$. However, expressions like $|x|^4\mathcal{P}_4(x)$ cannot be generated in ordinary perturbative quantum field theory, since c and f have perturbative expansions in powers of g and $\ln x\mu$. We shall see that the situation is crucially different in two dimensions.

The rest of this section is devoted to: i) work out the relationship between the two-point function and the trace anomaly; ii) discuss the two-dimensional case. We begin with the discussion on the trace anomaly.

Regulating the correlator (1.1) at $x = 0$ and taking the derivative with respect to $\ln \mu$, one gets

$$\frac{\mathrm{d}}{\mathrm{d}\ln \mu}\langle \mathcal{T}_{\mu\nu}(x)\, \mathcal{T}_{\rho\sigma}(0)\rangle \;=\; -\frac{1}{24\pi^2}\,\tilde{c}(g)\,\Pi_{\mu\nu\rho\sigma}\left(\delta^{(4)}(x)\right), \tag{1.9}$$

for a certain $\tilde{c}(g)$ that we now study. Any regularization technique is good, but we think that in this context it is particularly instructive to apply the differential technique of ref. [8], which is studied precisely for calculations or formal manipulations in the space of coordinates. To this purpose, let us write $c(g(x)) = \sum_n c_n(g)\, t^n$, $t = \ln x\mu$. Note that $c(g) = c_0(g)$. We observe that total μ-independence (with which we mean that g is differentiated as $\mathrm{d}g/\mathrm{d}\ln \mu = \beta(g)$) imposes $c_n'(g)\beta(g) + (n+1)\, c_{n+1}(g) = 0$. We can regulate according to a regularization formula that generalizes in a straightforward way the prescription of [8] and reads as follows

$$\frac{t^n}{x^4} \to -\frac{n!}{2^{n+1}}\Box\left(\sum_{k=0}^{n}\frac{2^k t^{k+1}}{(k+1)!}\frac{1}{x^2}\right) + a_n(g)\delta(x)\,, \tag{1.10}$$

the constants a_n parametrizing the scheme choice. The left and right hand sides of (1.10) differ just by $\delta(x)$-terms and effectively the regularization amounts to the factorization of the D'Alembertian [8]. Using (1.10) inside (1.8) and taking the total derivative with respect to $\ln \mu$ we get the derired result (1.9) after a number of simple manipulations. For example, with $a_n = 0$ one finds $\tilde{c}(g) = \sum_n n!\, c_n(g)/2^n$ and $c(g) = \tilde{c}(g) + \frac{1}{2}\,\beta(g)\,\mathrm{d}\tilde{c}(g)/\mathrm{d}g$: we see that the difference between $c(g)$ and $\tilde{c}(g)$ is proportional to the beta-function, so it is immaterial for the computation the IR limit of the central charge. It is easy to see that a different scheme choice adds another term proportional to the beta-function:

$$c(g) = \tilde{c}(g) + \frac{1}{2}\beta(g)\,\frac{\mathrm{d}}{\mathrm{d}g}(\tilde{c}(g) - s(g)), \qquad s(g) = \sum_n a_n(g)c_n(g)\,.$$

We conclude that off-criticality, only $c(g)$ is scheme independent, while $\tilde{c}(g)$ is not. At criticality $c(g) = \tilde{c}(g)$ and this is true also at two-loops around a free theory.

Eq. (1.9) is related to the integrated trace anomaly since the insertion of the operator $\int T$ corresponds precisely to taking the scale derivative $\mu\partial/\partial\mu$ [9]. One can check that the difference between $\mu\partial/\partial\mu$ and $\mathrm{d}/\mathrm{d}\ln \mu$ is still proportional to β (and so it vanishes at criticality), but it is now nonlocal. These observations illustrate the kind of problems that one typically finds when dealing, off-criticality, with the quantities appearing in the trace anomaly instead of the two- or four-point functions.

We conclude that the definition of central functions at the level of the trace anomaly equation is a more difficult task and it is preferable to pursue our objective at the level of correlators of conserved currents.

The direct relationship, at criticality, between the central charge c and the trace anomaly in external field supports the statement that c is invariant under continuous deformations of CFT_4's. Indeed, the anomaly of a current in external field is only one loop when all its potential internal anomalies vanish: all higher order rescattering graphs resum to zero in this case. In ref. [3] this property was explicitly shown to hold to two-loops, not only for c, but also for c', so that it was conjectured that the secondary central charge c' should also be invariant under continuous deformations of CFT_4. The idea is that c' is a generalization of the concept of anomaly in external field. Making this statement rigorous is still an open problem, but the two-loop results of [3] are a strong support to it (see next section and Fig. 1).

We finally stress the differences with the two-dimensional case, which has to be treated apart.

In two dimensions the denominator x^4 appearing in (1.1) is absent, the spin-two operator $\Pi_{\mu\nu\rho\sigma} = 2\pi_{\mu\nu}\pi_{\rho\sigma} - (\pi_{\mu\rho}\pi_{\nu\sigma} + \pi_{\mu\sigma}\pi_{\nu\rho})$ is identically zero and only the function f survives. Now, our considerations in higher dimension seem to exclude any role of f at criticality, where only c should survive. This apparent contraddiction is a sign that two dimensions are an exception: just note that f is not, according to our general definition, a good central function. We now point out the peculiar properties of this case and identify the true central function $c(g)$.

At criticality, $f = -c \ln x\mu$ [6]. Off-criticality f is a good function interpolating between the critical values of the ultraviolet and infrared central charges. Obviously, f cannot be written just as a function of g. This is not the effect of the mixing between the stress-energy tensor and an additional operator $\mathcal{S}_2$ (this problem is absent in two dimensions), rather it has to do with the problem of bypassing the forth-order differential operator $\pi_{\mu\nu}\pi_{\rho\sigma}$ when applying the Callan-Symanzik equations. In two dimensions the ambiguity generated in this process has the form $f \to f + \mathcal{P}_4(x)$. Lorentz invariance implies $\mathcal{P}_4(x) = a + bx^2$. However, the x^2-term cannot be generated in ordinary perturbation theory: this restricts $\mathcal{P}_4(x)$ to be a constant. Now, to overcome this constant, we consider the derivative $\dot{f}$ of f with respect to $t = \ln x\mu$. $\dot{f}$ satisfies the correct Callan-Symanzik equation: $\dot{f}(\lambda + t, g) = \dot{f}(t, g(\lambda))$. The most general solution is $f(t,g) = -\int_{t_0}^{t} c\,(g(t'))\,dt'$ for a certain $c(g)$ and an arbitrary t_0: this $c(g)$ is the desired central function. Therefore, (1.1) is replaced by

$$\langle T_{\mu\nu}(x)T_{\rho\sigma}(0)\rangle = \pi_{\mu\nu}\pi_{\rho\sigma}\left(-\int_{t_0}^{t} c\,(g(t'))\,dt'\right).$$

We now observe that reflection positivity of the correlator

$$\langle T(x)T(0)\rangle = \Box^2\left(-\int_{t_0}^{t} c\,(g(t'))\,dt'\right) = \Box\left(\frac{-\dot{c}}{x^2}\right) \geq 0$$

implies that $\dot{c} \leq 0$, i.e. that the function $c(g)$ satisfies the c-theorem [10]. Therefore, $c(g)$ is precisely the good function. These observations illustrate the peculiarities of two dimensions and show that our idea of central function is useful and consistent with the notions introduced in the previous investigations.

2. Definition of the secondary central functions

The correlator $\langle \mathcal{T}_{\mu\nu}(x)\mathcal{T}_{\rho\sigma}(y)\mathcal{T}_{\alpha\beta}(z)\mathcal{T}_{\gamma\delta}(w)\rangle$ generates the secondary central functions c'. The discussion of the previous paragraph can be seen as a particular case of the more involved construction to be presented now. In particular, the primary central function is a very special secondary central function.

Generically, a secondary central function $c'_{\mathcal{O}}(g)$ is associated with any operator $\mathcal{O}$, the most relevant cases being $\mathcal{O}=J_\mu^5$ and $\mathcal{O}=\bar{\varphi}\varphi$. In the simplest situation, one can assume that $\mathcal{O}$ does not mix with other operators under renormalization. More generically, the definition that we are going to give applies to a *set* of operators $\{\mathcal{O}_i\}$ that is closed under renormalization mixing and irreducible in this sense. For the moment we assume that the $\{\mathcal{O}_i\}$ have the same canonical dimension d.

For example, in ref. [3] the operators $\{\mathcal{O}_i\}$ are the various Konishi superfields K_i associated with the different irreducible representations of the matter multiplets. The K_i mix at the quantum level and this prevents us from constructing a function c' for each of them: instead, the K_i's contribute to the *same* central function c', unless some flavour symmetry or similar property allows one to split the set of Konishi superfields into different irreducible subsets $\{K_i\}_1,\ldots\{K_i\}_n$, each of which will have its own c'-function $c'_1,\ldots c'_n$.

In this paragraph we assume that the $\mathcal{O}_i$ are scalar operators. This simplifies the formulæ considerably and is sufficient to illustrate how to prodeed in generality.

Let $Z_{ij}(\ln x^2\mu^2)$ denote the renormalization constants of the operators $\mathcal{O}_i$. The correlators

$$\langle \mathcal{O}_i(x)\mathcal{O}_j(0)\rangle = \frac{1}{x^{2d}}Z_{ik}(\ln x\mu, g)A_{kl}(g(x))Z_{jl}(\ln x\mu, g) \tag{2.1}$$

define functions $A_{ij}(g)$ that depend only on g. The proof is a straightforward application of the Callan-Symanzik equations (1.5). Similarly, the OPE

$$\mathcal{T}_{\mu\nu}(x)\mathcal{T}_{\rho\sigma}(0) \sim \frac{1}{144\pi^2}\Pi_{\mu\nu\rho\sigma}\left(\frac{B_i(g(x))Z^{-1}_{ij}(\ln x\mu, g)}{x^{4-d}}\right)\mathcal{O}_j(0) \tag{2.2}$$

defines other functions $B_i(g)$ that share the same property. We used directly $\mathcal{T}_{\mu\nu}$ in (2.2) rather than $T_{\mu\nu}$, to simplify the derivation: it is understood that an analysis similar to the one carried out for the primary central function can be repeated starting from $T_{\mu\nu}$. We define

$$c'_{\mathcal{O}}(g) \equiv B_i(g)A_{ij}(g)B_j(g)\,. \tag{2.3}$$

This function interpolates between the appropriate critical values and coincides with the function computed in ref. [3] to the second loop order included.

We stress that formulæ (2.1), (2.2) and (2.3) have a nontrivial meaning also at criticality, if the matter fields are in a reducible representation of the gauge group. Then, $Z_{ij} = |x\mu|^{h_{ij}(g)}$, $h_{ij}(g)$ being a matrix that, in general, is not symmetric. It is simple to work out examples of this kind starting from the results of [3].

We note that c' is scheme independent, while the quantities $A_{ij}(g)$ and $B_i(g)$ do depend on the subtraction scheme. To be precise, compatibility with the Callan-Symanzik equations requires that a scheme change dictated by $\mathcal{O}_i \rightarrow H_{ij}(g)\mathcal{O}_j$ be expressed as $Z(\ln x\mu, g) \rightarrow H(g)Z(\ln x\mu, g)H^{-1}(g(x))$, $A(g) \rightarrow H(g)A(g)H^t(g)$, $B(g) \rightarrow B(g)H^{-1}(g)$. Consequently, c' is invariant, because the finite subtractions simplify in expression (2.3): $c'(g) \rightarrow c'(g)$. This observation shows, to all orders in perturbation theory, off-criticality and in the case of nontrivial renormalization mixing between operators, that the function $c'(g)$ is a good central function.

It is straightforward to extend the definition of $c'(g)$ to the case of mixing among operators of different canonical dimensions. This case is not of primary interest for the applications and is left to the reader.

With a suitable combination of limiting procedures, it is possible to single out the secondary central functions inside the correlator $\langle \mathcal{T}_{\mu\nu}(x)\mathcal{T}_{\rho\sigma}(y)\mathcal{T}_{\alpha\beta}(z)\mathcal{T}_{\gamma\delta}(w)\rangle$. It is certainly true that the correlator of four stress-tensors has no unique expression. Nevertheless in the limit in which the first couple of stress tensors is very far from the other couple then the form of the correlator becomes unique, up to an overall constant at criticality (the secondary central charge) or up to a central function off-criticality.

Roughly, one has to "open" the four point correlator $\langle \mathcal{T}_{\mu\nu}(x)\mathcal{T}_{\rho\sigma}(y)\mathcal{T}_{\alpha\beta}(z)\mathcal{T}_{\gamma\delta}(w)\rangle$ and look carefully inside. One first takes the limit in which two distances, for example $\delta_1 \equiv |x-y|$ and $\delta_3 \equiv |z-w|$, are much smaller than the other one, $\Delta_2 \equiv |x-w|$. This procedure factorizes two differential operators $\Pi_{\mu\nu\rho\sigma}$ and $\Pi_{\alpha\beta\gamma\delta}$, acting on a certain expression $\mathcal{E}(\delta_1, \Delta_2, \delta_3) = \sum_{\mathcal{O}} \mathcal{E}_{\mathcal{O}}(\delta_1, \Delta_2, \delta_3)$, with canonical powers distributed among the distances δ_1, Δ_2 and δ_3 according to the canonical dimension of the intermediate channel $\mathcal{O}$. Precisely,

$$\begin{aligned} \langle \mathcal{T}_{\mu\nu}(x)\, \mathcal{T}_{\rho\sigma}(y)\, \mathcal{T}_{\alpha\beta}(z)\, \mathcal{T}_{\gamma\delta}(w)\rangle|_{\delta_{1,3}\ll\Delta_2} &\sim \frac{1}{(144\pi^2)^2} \sum_{\mathcal{O}} \overrightarrow{\Pi}^{(1)}_{\mu\nu\rho\sigma} \left(\frac{B_i(g(\delta_1))Z^{-1}_{ij}(\delta_1, g)}{\delta_1^{4-d}} \times \right. \\ &\times \left. \frac{Z_{jk}(\Delta_2, g)A_{kl}(g(\Delta_2))Z_{ml}(\Delta_2, g)}{\Delta_2^{2d}} \, \frac{Z^{-1}_{nm}(\delta_3, g)B_n(g(\delta_3))}{\delta_3^{4-d}} \right) \overleftarrow{\Pi}^{(3)}_{\alpha\beta\gamma\delta} \, . \end{aligned} \quad (2.4)$$

Then, taking the limit $\delta_1 = \Delta_2 = \delta_3 = \lambda$, one gets $\mathcal{E}_{\mathcal{O}}(\lambda) = c'_{\mathcal{O}}(g(\lambda))/\lambda^8$. The construction is illustrated in Fig. 1.

In CFT_2, the only channels $\mathcal{O}$ are the stress tensor itself and its derivatives,[3] so one gets the usual central charge. In higher dimensions, instead, one has infinitely many channels $\mathcal{O}$ and central charges (or functions, off-criticality). Consequently, the basic correlator that one has to consider in order to properly describe higher dimensional CFT's or quantum field theories interpolating among them is precisely the correlator of four stress tensors.

[3] The identity operator, on the other hand, cannot be considered as an intermediate channel, since it contributes only to disconnected diagrams.

JHEP05(1998)005

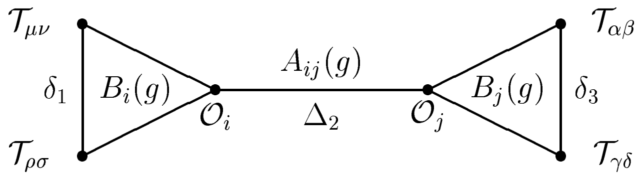

Figure 1: Construction of $c'(g) = B_i(g)A_{ij}(g)B_j(g)$. It is argued that this graph represents the generalization of the famous triangle graph of the ordinary notion of anomaly.

It is true in any dimension that the secondary central function associated with the channel $\mathcal{O} = \mathcal{T}_{\mu\nu}$ coincides with the primary central function itself: $c'_{\mathcal{T}_{\mu\nu}}(g) = c(g)$. This is because $B = 1$, $A(g) = c(g)$ in this very special case.

To conclude this section, let us summarize the idea behind our construction. At first, one might think that at criticality it is consistent to associate a central charge to any operator $\mathcal{O}(x)$, i.e. the coefficient of the identity operator in the OPE $\mathcal{O}(x)\mathcal{O}(0)$. However, this is not the case, as one can immediately see by going off-criticality: no good central function can be generated by a correlator of non-conserved currents or operators of a different nature. First of all, one has to eliminate the Z-factors, otherwise one does not define just a function of $g(x)$, but a function of $g(x)$ and x separately: see, for example, the function f of eq. (2.1). Even after getting rid of the Z-factors, $A(g)$ turns out to be a scheme dependent quantity, as discussed below eq. (2.3). Moreover, due to operator mixing, the quantity $A_{ij}(g)$ is in general a matrix and not a function. Finding appropriate B_i's to paste to it while preserving a physical meaning is not trivial. For this reason, just those operators $\mathcal{O}_i$ that appear in the stress-tensor OPE are considered here and their appearance in the stress-tensor OPE automatically defines the appropriate B_i's. It is clear, then, that the relevant correlator is the one of four stress-tensors.

Finally, we note that most observations cannot be worked out while staying at criticality, yet they have relevant consequences on CFT_4: for example, we have just remarked that only those quantities that can be promoted to central functions off-criticality deserve to be called central charges at criticality. Therefore, it is thanks to the construction presented here that the notion of secondary central charges used for c' in [1, 3] is firmly justified.

If the quantum field theory under consideration admits other conserved currents J (for example flavour currents), the construction presented here can be generalized in an obvious way to the correlators $\langle JJ\rangle$ and $\langle JJJJ\rangle$. J replaces $\mathcal{T}$ and defines "flavour" primary and secondary central charges.

Finally, another noticeable central charge to describe a CFT_4 is the coefficient a of the Euler density in the trace anomaly of the energy-momentum tensor in the external gravitational field,

$$T = T_{\mu\mu} \sim c\, W^2 \,+\, a \text{ Euler} \,+\, \cdots$$

W denoting the Weyl tensor and the dots standing for contributions of external fields other than the gravitational one and internal fields. Since the Euler density a total derivative, the quantity a is not visible in (1.9), because the scale derivative $\mu\,\partial/\partial\mu$ is equivalent to the insertion of the integrated trace anomaly $\int T$. It is important to remark that we do not possess, yet, a good definition of central function $a(g)$ that interpolates between the critical values a_{UV} and a_{IR} of the central charge a. Finding out this function would be extremely interesting, since it is expected to monotonically decrease from the UV to the IR (see [11]).

3. Comments and conclusions

In this paper, we have worked out all-order definitions of certain central functions that might be useful to extract relevant properties of quantum field theories. The construction is non-trivial, but it is also quite general, natural and unique. Not all the quantities of CFT_4 can be promoted to central functions off-criticality and several functions are not "good" central functions. We are carefully identified the quantities that have the good properties. The idea is that if some quantities will ever be computed exactly down to the far infrared of QCD, they will be presumably some of the functions constructed here. If there is a c-theorem in four dimensions, our construction might be relevant to work out a proof.

Our quantities can be studied in the context of perturbation theory. The first nontrivial correction appears at the second loop order and measures the slope $\mathrm{d}c/\mathrm{d}\alpha$ at criticality. Nevertheless, the aim of the investigation that we have addressed in this paper is to derive nonperturbative expressions of the central functions. We do not have a general method to achieve this goal for the moment. This task might be very difficult, but the properties singled out in [3] and here make us hope that it is affordable. After completion of this work, moreover, the problem was solved in supersymmetric theories for the quantities c, a and for the flavour central charges [11]. In more generality, non-perturbative ideas might allow us to extend the results of [11] to the secondary central functions. It is clear that doing something similar for nonsupersymmetric theories would be extremely fruitful for the understanding of the low-energy limit of quantum gauge field theories.

Acknowledgments

I am grateful to U. Aglietti, D.Z. Freedman and A.A. Johansen for useful correspondence and discussions. This work is partially supported by EEC grants CHRX-CT93-0340 and TMR-516055.

References

[1] D. Anselmi, M.T. Grisaru and A.A. Johansen, *Anomalous currents, electric-magnetic universality and* CFT_4, *Nucl. Phys.* **B 491** (1997) 221 [hep-th/9601023].

[2] Discussions about the nonclosure of the stress-tensor OPE's in $d > 2$ can be also found in the previous literature. See, for example, Y. Stanev, *Bulgarian Journal of Physics* **15** (1988) 93, in $d = 4$ and A.C. Petkou, *Ann. Phys. (NY)* **249** (1996) 180 in the $O(N)$ vector model in $2 < d < 4$.

[3] D. Anselmi, D.Z. Freedman, M.T. Grisaru and A.A Johansen, *Universality of the operator product expansions in* $SCFT_4$, *Phys. Lett.* **B 394** (1997) 329 [hep-th/9608125].

[4] V. Novikov, M.A. Shifman, A.I. Vainshtein and V. Zakharov, *Exact Gell-Mann-Low Function of Supersymmetric Yang-Mills Theories from Instanton Calculus*, *Nucl. Phys.* **B 229** (1983) 381;
M.A. Shifman and A.I. Vainshtein, *Solution of the Anomaly Puzzle in supersymmetric Gauge Theories and the Wilson Operator Expansion*, *ibid.* **277** (1986) 456.

[5] N. Seiberg, *Electric-Magnetic Duality in Supersymmetric Non-Abelian Gauge Theories*, *Nucl. Phys.* **B 435** (1995) 129 [hep-th/9411149].

[6] A. Cappelli, D. Friedan and J.I. Latorre, *c-Theorem and Spectral Representation*, *Nucl. Phys.* **B 352** (1991) 616.

[7] See for example J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, Oxford Science Publications, 1993, p. 296.

[8] D.Z. Freedman. K. Johnson and J.I. Latorre, *Differential Regularization and Renormalization: a New Method of Calculation in Quantum Field Theory*, *Nucl. Phys.* **B 371** (1992) 353.

[9] S.L. Adler, J.C. Collins and A. Duncan, *Energy-Momentum Tensor Trace Anomaly in Spin-1/2 Quantum Electrodynamics*, *Phys. Rev.* **D 15** (1977) 1712.

[10] A.B. Zamolodchikov, *"Irreversibility" of the Flux of the Renormalization Group in a 2D Field Theory*, *JETP Lett.* **43** (1986) 730.

[11] D. Anselmi, D.Z. Freedman, M.T. Grisaru and A.A. Johansen, *Nonperturbative formulas for central functions in supersymmetric gauge theories*, hep-th/9708042.